This exam consists of **three problems**, worth a total of 40 points.

Problem 1 has one part, and is worth 15 points.
Problem 2 has three parts, and is worth 15 points.
Problem 3 has one part, and is worth 10 points.

Below are a few preliminaries and definitions that you may find useful for the exam.

- For an N-point discrete-time sequence $x[n]$ with $x[n] = 0$ for $n < 0$ and $n > N-1$, we denote its N-point discrete Fourier Transform (DFT) by $X[k]$, where $X[k] = 0$ for $k < 0$ and $k > N-1$. The analysis and synthesis equations are:

$$\text{Analysis equation:} \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad 0 \le k \le N-1,$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad 0 \le n \le N-1,$$

where $W_N \triangleq e^{-j(2\pi/N)}$.

- For a general discrete time sequence $x[n]$, we denote its discrete-time Fourier Transform (DTFT), when it exists, by $X(e^{j\omega})$. The analysis and synthesis equations are:

$$\text{Analysis equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega.$$

Similarly, the z-transform of a discrete-time signal $x[n]$ is given by $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$.

1) **[15 points]** Let $X(e^{j\omega})$ be the discrete time Fourier Transform (DTFT) of the discrete-time signal $x[n] = (1/2)^n u[n]$. Find a length-5 sequence $g[n]$ whose five-point discrete Fourier transform (DFT) $G[k]$ is identical to the samples of the DTFT of $x[n]$ at $\omega_k = 2\pi k/5$, i.e.,

$$g[n] = 0 \quad \text{for} \quad n < 0,\ n > 4$$

and

$$G[k] = X(e^{j2\pi k/5}) \quad \text{for} \quad k = 0, 1, \ldots, 4.$$

Solution: The 5 samples of $X(e^{j\omega})$ are effectively obtained by aliasing $x[n]$ to $0 \leq n < 5$ and taking a DFT. For example, the aliased signal $g[0]$ will be the sum of the elements $x[0 + 5m]$ over all (positive and negative) integers m. More generally, for $n = 0, 1, \ldots, 4$, we have

$$\begin{aligned}
g[n] &= \sum_{m=-\infty}^{\infty} x[n+5m] \\
&= \sum_{m=0}^{\infty} x[n+5m] \qquad \text{(since } x \text{ is causal)} \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{n+5m} = \left(\frac{1}{2}\right)^{n} \sum_{m=0}^{\infty} \left(\frac{1}{32}\right)^{m} = \left(\frac{1}{2}\right)^{n} \left(\frac{1}{1-1/32}\right) = \left(\frac{32}{31}\right) \left(\frac{1}{2}\right)^{n}.
\end{aligned}$$

Note that we can obtain this general formula using the DFT synthesis equation directly:

$$\begin{aligned}
g[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j2\pi k/N})\, e^{j2\pi kn/N} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi km/N} \right) e^{j2\pi kn/N} \\
&= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{k=0}^{N-1} \left(\frac{1}{N}\right) e^{j2\pi k(n-m)/N} \right) \\
&= \sum_{m=-\infty}^{\infty} x[m] \sum_{r=-\infty}^{\infty} \delta[n-m+rN] \quad \text{(the second sum is the FS rep'n of an impulse train)} \\
&= \sum_{r=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m]\delta[n+rN-m] \right) = \sum_{r=-\infty}^{\infty} x[n+rN]
\end{aligned}$$

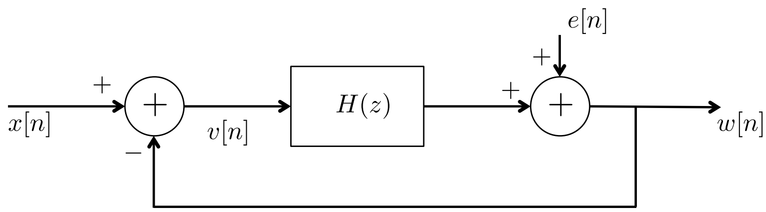
which is essentially the same as the expression above.

An alternative approach would use the DTFT of $x[n]$ directly:

$$g[n] = \frac{1}{5} \sum_{k=0}^{N-1} \underbrace{\left(\frac{1}{1-(1/2)e^{-j2\pi k/5}} \right)}_{X(e^{j\omega})|_{\omega=2k\pi/5}} e^{j2\pi kn/5}$$

but this could be cumbersome to evaluate.

2) **[15 points total, 3 parts]** In the figure below, $H(z)$ is the system function of a causal linear time invariant (LTI) system.



a) **[5 points]** Using z-transforms of the signals shown in the figure, obtain an expression for $W(z)$ in the form

$$W(z) = H_1(z)X(z) + H_2(z)E(z),$$

where both $H_1(z)$ and $H_2(z)$ are expressed in terms of $H(z)$.

Solution: It is easy to see that

$$\begin{aligned} V(z) &= X(z) - W(z) \\ W(z) &= V(z)H(z) + E(z), \end{aligned}$$

so solving for $W(z)$ gives

$$W(z) = \underbrace{\left(\frac{H(z)}{1+H(z)}\right)}_{H_1(z)} X(z) + \underbrace{\left(\frac{1}{1+H(z)}\right)}_{H_2(z)} E(z).$$

b) **[5 points]** For the special case $H(z) = z^{-1}/(1-z^{-1})$, determine $H_1(z)$ and $H_2(z)$.

Solution: By straightforward calculation,

$$H_1(z) = \frac{\frac{z^{-1}}{1-z^{-1}}}{1+\frac{z^{-1}}{1-z^{-1}}} = \frac{z^{-1}}{1-z^{-1}+z^{-1}} = z^{-1}, \quad H_2(z) = \frac{1}{1+\frac{z^{-1}}{1-z^{-1}}} = 1 - z^{-1}.$$

c) **[5 points]** Is the system $H(z)$ stable? Are the systems $H_1(z)$ and $H_2(z)$ stable?

Solution: The system $H(z)$ is not stable because it has a pole at $z = 1$. But, the systems $H_1(z)$ and $H_2(z)$ are each stable – indeed, it is easy to see that both $H_1(z)$ and $H_2(z)$ have impulse responses of finite duration:

$$h_1[n] = \begin{cases} 1, & n = 1 \\ 0, & \text{otherwise} \end{cases} \qquad h_2[n] = \begin{cases} 1, & n = 0 \\ -1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

and their ROC's contain the entire z-plane except for $z = 0$.

3) **[10 points]** Suppose $x[n]$ is a finite-duration discrete-time signal that is known to be zero for all $n < 0$.

Show that if $x[n]$ is also *binary-valued* (i.e., its nonzero components can only take the value 1) it can be recovered exactly from its z-transform evaluated at the *single point* $z = 1/2$. That is, provide a simple strategy for recovering causal, binary-valued signals $x[n]$ from $X(z)|_{z=1/2}$.

Solution: We begin by writing the expression for $X(1/2)$:

$$\begin{aligned} X(1/2) &= \sum_{n=-\infty}^{\infty} x[n](1/2)^{-n} \\ &= \sum_{n=0}^{\infty} x[n]2^{n} \quad \text{(since } x \text{ is causal).} \end{aligned}$$

Now, if $x[n]$ is also binary-valued, then the sum on the right-hand side is just a sum of integer powers of 2, each with weight 0 or 1. But, this is just the *decimal representation* of the number represented by the binary expansion $\ldots x[4] \ldots x[3] \ldots x[2] \ldots x[1] \ldots x[0]$. Further, the assumption that $x[n]$ be of finite duration guarantees that this number will be finite (so that $z = 1/2$ is in the ROC).

Thus, $x[n]$ can be reconstructed as follows. First, find the binary representation of the (real, integer) value $X(1/2)$. Then, impute the values $x[0], x[1], \ldots$ from the binary expansion starting with the least-significant bit.